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Computations over $\mathbb{Z}$ and $\mathbb{R}$: A Comparison

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Regarding real number models of computation may be a helpful way to get deeper insight into the classical theory over $\mathbb{Z}$. Therefore it seems useful to study the complexity of classical problems in the real model. In this connection, the problem of deciding the existence of a nonnegative zero for certain polynomials plays an important part because a lot of NP-problems over $\mathbb{Z}$ can be polynomial reduced to it. 1990 Academic Press, Inc.

1. INTRODUCTION

Blum, Shub, and Smale (1989) have introduced a model of computation over the real numbers together with a theory of complexity. A problem which arises at once is the question for the complexity of NP-problems over $\mathbb{Z}$ with regard to this new model.

In Section 2, Theorem 1 shows that the 3-SAT problem, which is well known to be NP-complete over $\mathbb{Z}$ (see Garey and Johnson, 1979), can be polynomially reduced to the problem $(F^2, F^2_{\text{zero},+})$ in the new theory (i.e., the problem of whether a polynomial f: $\mathbb{R}^n \to \mathbb{R}$ of degree 2 has a nonnegative zero or not). Subsequently we prove that the latter reducibility is not valid for all decision problems belonging to NP over $\mathbb{Z}$. So, reducibility over $\mathbb{Z}$ does not imply reducibility over $\mathbb{R}$.

Section 3 deals with the $NP_{\mathbb{R}}$-completeness of $(F^4, F^4_{\text{zero},+})$ in the sense of Blum *et al.*, 1989.

2. REDUCIBILITY OF (3-SAT,3-SAT_{yes}) TO $(F^2, F^2_{\text{zero},+})$: REPRESENTATION FOR SPECIAL DECISION PROBLEMS

Before going into details we introduce some notation: given sets Y and $L \subseteq Y$, (Y, L) denotes the decision problem whether $y \in Y$ belongs to L or

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not. By $P_{\mathbb{Z}}$ and $NP_{\mathbb{Z}}$ we mean the classical complexity classes of deterministic and nondeterministic polynomial time-computable decision problems; $P_{\mathbb{R}}$ and $NP_{\mathbb{R}}$ describe the analogous classes in the Blum–Shub–Smale (BSS) model.

Similarly:

—$(Y_1, L_1) \propto_{\mathbb{Z}} (Y_2, L_2)$ and $(Y_1, L_1) \propto_{\mathbb{R}} (Y_2, L_2)$ denote polynomial time reducibility of (Y_1, L_1) to (Y_2, L_2) over $\mathbb{Z}$ and $\mathbb{R}$, respectively.

—$\text{size}_{\mathbb{Z}}(y)$ and $\text{size}_{\mathbb{R}}(y)$ describe the sizes of y in the different models.

For any $i \in \mathbb{N}$, F^i is the set of all polynomials f with f: $\mathbb{R}^n \to \mathbb{R}$ for some $n \in \mathbb{N}$, degree of $f \leq i$; $F^i_{\text{zero}} \subseteq F^i$ are those elements of F^i which have a real zero, and $F^i_{\text{zero},+} \subseteq F^i$ are those having a nonnegative real zero (i.e., $\exists$ $x_1, \ldots, x_n$ such that $f(x_1, \ldots, x_n) = 0$ and $x_j \geq 0$, $1 \leq j \leq n$).

Finally,

k-SAT := $\{\Phi \mid \Phi$ is a conjunctive-normal-form formula with at most k literals per clause$\}$ (cf. Garey and Johnson, 1979)

k-SAT$_{\text{yes}}$:= k-SAT $\cap$ $\{\Phi \mid \Phi$ is satisfiable$\}$

(Φ is called "satisfiable" iff there exist $x_1, \ldots, x_n$ such that $\Phi(x_1, \ldots, x_n) = 1$).

THEOREM 1. *(3-SAT, 3-SAT$_{\text{yes}}$)* $\propto_{\mathbb{R}} (F^2, F^2_{\text{zero},+})$.

Proof. Let Φ be a 3-SAT formula over the variables $x_1, \ldots, x_n$, say $\Phi(x_1, \ldots, x_n) = C_1 \cdot C_2 \cdots C_m$ for some $m \in \mathbb{N}$, $C_i = x_{i,1}^{\{\alpha_i\}} \vee x_{i,2}^{\{\beta_i\}} \vee x_{i,3}^{\{\gamma_i\}}$ $(x_{i,j} \in \{x_1, \ldots, x_n\}\ \forall\ 1 \leq i \leq n, 1 \leq j \leq 3, \alpha_i, \beta_i, \gamma_i \in \{0, 1\}$ and $x_{i,j}^{\{0\}} := \overline{x_{i,j}}$, $x_{i,j}^{\{1\}} := x_{i,j})$.

Define

$$y_{i,1} := \begin{cases} (1 - x_{i,1})^2 & \text{if } \alpha_i = 1 \\ x_{i,1} & \text{if } \alpha_i = 0, \end{cases}$$

similarly for $y_{i,2}$ and $y_{i,3}$.

For each clause C_i introduce three new variables $\lambda_{3i-2}, \lambda_{3i-1}, \lambda_{3i}$ and let

$$\left.\begin{aligned} &\tilde{p}_i(x_{i,1}; x_{i,2}; x_{i,3}; \lambda_{3i-2}; \lambda_{3i-1}; \lambda_{3i}) := y_{i,1}\lambda_{3i-2} + y_{i,2}\lambda_{3i-1} + y_{i,3}\lambda_{3i} \\ &\qquad\qquad + (1 - (\lambda_{3i-2} + \lambda_{3i-1} + \lambda_{3i}))^2 \\ &\text{and} \\ &\tilde{p}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{3m}) := \sum_{i=1}^{m} \tilde{p}_i(x_{i,1}; \ldots; \lambda_{3i}). \end{aligned}\right\} \quad (*)$$

If all x_k and λ_l are ≥ 0 then $\tilde{p}_i(x_{i,1}; \ldots; \lambda_{3i}) \geq 0$ and hence $\tilde{p}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{3m}) \geq 0$. Therefore every nonnegative zero of $\tilde{p}$ has to be a (nonnegative) zero of $\tilde{p}_i$, $1 \leq i \leq m$. Let $(x_{i,1}; \ldots ; \lambda_{3i})$ be a nonnegative zero of $\tilde{p}_i$; it follows that each term of the sum in $\tilde{p}_i$ vanishes. Hence, at least one of the unknowns λ_{3i-2}, λ_{3i-1}, and λ_{3i} is positive, so the corresponding $y_{i,j}$ must be 0. But this implies $x_{i,j}$ to satisfy the clause C_i. Consequently, every nonnegative zero of $\tilde{p}$ gives a truth assignment of Φ. (Of course, if $\tilde{p}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{3m}) = 0, x_k, \lambda_l \geq 0$, not every x_k has to be 0 or 1; nevertheless those x_k which are forced to belong to $\{0, 1\}$ in virtue of a nonvanishing λ_k still satisfy Φ. Because of this, the other x_k can be chosen to be 0 without affecting $\Phi(x_1, \ldots, x_n) = 1$.) Conversely, if $\Phi(x_1, \ldots, x_n) = 1$, for each C_i there is a $z_i \in \{x_1, \ldots, x_n\}$ which satisfies C_i. Let the λ value corresponding to this particular z_i in (*) be 1, the other ones $= 0$; this yields a nonnegative zero of $\tilde{p}$.

At last we have to remove the degree-3 terms, which may appear in $\tilde{p}_i$ if $y_{i,j} = (1 - x_{i,j})^2$, i.e., exactly those $y_{i,j}$, for which $x_{i,j}$ is not negated in C_i. Let $A \subseteq \{x_i, \ldots, x_n\}$ be the set of these variables. For each $x_k \in A$ let u_k be a further unknown and replace each occurrence of $(1 - x_k)^2$ in $\tilde{p}$, for example, $(1 - x_{i,j})^2 \cdot \lambda_{3i-3+j}$ by $u_{i,j} \cdot \lambda_{3i-3+j}$.

Finally, add the term $\sum_{x_k \in A} (1 - x_k - u_k)^2$ and denote the arising polynomial by p,

$$p(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{3m}, u_{i_1}, \ldots, u_{i_s}) \qquad (s := |A|, x_{i_j} \in A \ \forall 1 \leq j \leq s).$$

We now have

—degree of $p = 2$

—$\exists$ nonnegative zero for $p \Leftrightarrow \exists$ nonnegative zero for $\tilde{p} \Leftrightarrow \Phi$ is satisfiable

—p is polynomial time constructible for given Φ since $\text{size}_{\mathbb{R}}(\Phi) = O(m)$ (we have introduced at most $3m + n$ new variables; note that, in a "convenient" representation of Φ, n is bounded by $3m$).

We omit the explicit construction of a Blum–Shub–Smale machine in class $P_{\mathbb{R}}$ which establishes the reduction. ■

Remark. One also has (k-SAT, k-SAT$_{\text{yes}}$) $\propto_{\mathbb{R}} (F^2, F^2_{\text{zero},+})$ because the reduction (k-SAT, k-SAT$_{\text{yes}}$) $\propto_{\mathbb{Z}}$ (3-SAT, 3-SAT$_{\text{yes}}$) can be done in the same way over $\mathbb{R}$. This mainly depends on the fact that $\text{size}_{\mathbb{Z}}(\Phi)$ and $\text{size}_{\mathbb{R}}(\Phi)$ are polynomially related for any $\Phi \in$ k-SAT.

However, the latter is not the case in general.

In Blum *et al.* (1989, Sect. 1, Example 5; Sect. 4, Proposition 3) there is given a lower bound on the halting time for computing the "greatest integer in." The basic idea hidden in this example is formalized in the following theorem.

THEOREM 2. *Let (Y, L) be a decision problem, $L \subseteq Y \subseteq \mathbb{R}$. Then we have $(Y, L) \in P_{\mathbb{R}}$ if and only if $L = Y \cap \tilde{M}$, where $\tilde{M}$ is a finite union of intervals in $\mathbb{R}$.*

Proof. The "*only if*" part: Assume $(Y, L) \in P_{\mathbb{R}}$, hence $\exists$ a Blum–Shub–Smale machine M with admissible inputs $y \in Y$ and a polynomial p such that

$$\psi_M(y) := \begin{cases} 1, & y \in L \\ 0, & y \in Y \setminus L \end{cases} \qquad \text{(where } \psi_M \text{ is the function computed by } M\text{)}$$

and the halting time $T_M(y)$ is bounded by $p(\text{size}_{\mathbb{R}}(y)) = p(1) =: T$ for all $y \in Y$ (in Blum *et al.* (1989) a real number y has $\text{size}_{\mathbb{R}}(y) = 1$, so T is independent of y).

M may have a labeling $\{1, \ldots, N\}$ (1 labels the input-node and N labels the output-node). A "path" of M is a finite sequence $\gamma = \gamma_1, \ldots, \gamma_T \in \{1, \ldots, N\}^T$, where $\gamma_1 = 1$, $\gamma_T = N$; for $y \in Y$, $\gamma(y)$ denotes the path passed by M while $\psi_M(y)$ is calculated. Let $\Pi_1, \ldots, \Pi_m$ be the finitely many paths of M ($m \leq 2^T$) and define

$$V_i := \{x \in Y \mid \gamma(x) = \Pi_i\}, \qquad 1 \leq i \leq m.$$

Now let i be fixed; at every node of the path Π_i either polynomial calculations or polynomial tests, both depending only on the input x, are performed. We restrict ourselves to the test functions, say "$h_{i,j}(x) \geq 0$?" $1 \leq j \leq k_i$. Each of these polynomials $h_{i,j}$ divides Y into two sets

$$\text{(i)} \quad Y \cap A_{i,j} \qquad \text{and} \qquad \text{(ii)} \quad y \cap B_{i,j},$$

where $A_{i,j} = \{(x \in \mathbb{R} \mid h_{i,j}(x) \geq 0\}$ is a finite union of closed intervals and $B_{i,j} = \{x \in \mathbb{R} \mid h_{i,j}(x) \leq 0\}$ is a finite union of open intervals. Hence, we get V_i as the intersection of a finite number of sets having a representation of the form (i) or (ii). Consequently, we have

$$V_i = Y \cap M_i, \qquad \text{where } M_i \text{ is a finite union of intervals} \qquad (**)$$

(observe that $\bigcup_{i=1}^{n} V_i = Y$, since for each input $x \in Y$, M halts).

We still must show that a similar decomposition holds for L: without loss of generality, let $V_1, \ldots, V_n$ be those V_j having nonempty intersection with L; $V_1, \ldots, V_s$, $s \leq n$, denotes those having finite cardinality. Then

$$L = \bigcup_{i=1}^{n} (L \cap V_i) = \left(\bigcup_{i=1}^{s} L \cap V_i\right) \cup \left(\bigcup_{i=s+1}^{n} V_i\right)$$

(note that ψ_M restricted to V_i is a polynomial which takes the value 1 for $x \in L$ and 0 for $x \in Y \setminus L$. Therefore if V_i has infinite cardinality (i.e., $s + 1 \le i \le n$), $\psi_M|_{V_i}$ must be constant $=1$ because $V_i \cap L \ne \emptyset$. But $\Psi_M|_{V_i} = 1$ implies $V_i \subset L$).

For $1 \le i \le s$, let $A_i = L \cap V_i$ (note that A_i is finite) and define $\tilde{M} := (\bigcup_{i=1}^{s} A_i) \cup \bigcup_{i=s+1}^{n} M_i)$. Then, $\tilde{M}$ is a finite union of intervals (see (**)) and $L = Y \cap \tilde{M}$.

The *"if"* part: If L has a representation of the required form, for each input $x \in Y$, a BSS machine must check whether or not x belongs to one of the finitely many intervals of $\tilde{M}$. This can be done by polynomial tests "$h(x) \ge 0$?" in constant time; hence $(Y, L) \in P_{\mathbb{R}}$. ■

Remarks.

—It follows at once: the decision problems $(\mathbb{R}, \mathbb{Z})$, $(\mathbb{R}, \mathbb{Q})$, $(\mathbb{R}, \mathbb{N})$, and $(\mathbb{Q}, \mathbb{Z})$ do not belong to $P_{\mathbb{R}}$. (Note that $(\mathbb{R}, \mathbb{Q})$ is not even decidable in the BSS model. This follows from a simular argument used in Theorem 2.)

—In Theorem 2 we can replace $P_{\mathbb{R}}$ by $NP_{\mathbb{R}}$ because the two classes coincide if they are restricted to finite-dimensional decision problems (i.e., problems (Y, L), $Y \subseteq \mathbb{R}^m$; if $(Y, L) \in NP_{\mathbb{R}}$ we can reduce it to (F^4, F^4_{zero}) by a polynomial time-computable function ψ: $Y \to F^4$. Since $\text{size}_{\mathbb{R}}(y) \le m \ \forall\, y \in Y$, $\text{size}_{\mathbb{R}}(\psi(y)) =: T$ is independent of y and one can decide $\psi(y) \in F^4_{\text{zero}}$? in constant time (see Tarski, 1951; for explicit upper bounds confer Renegar (1988)).

We now have the

COROLLARY. *The problem QDE (i.e., quadratic diophantine equations: given $a, b, c \in \mathbb{N}$, do there exist $x, y \in \mathbb{N}$ s.t. $ax^2 + by - c = 0$) is not in $NP_{\mathbb{R}}$. Hence $(Y_1, L_1) \propto_{\mathbb{Z}} (Y_2, L_2)$ does not imply $(Y_1, L_1) \propto_{\mathbb{R}} (Y_2, L_2)$ in general.* (QDE obviously belongs to $NP_{\mathbb{Z}}$, so it is $\mathbb{Z}$-reducible to 3-SAT; a nondeterm. The BSS machine solving QDE at first must guess $x, y \in \mathbb{R}$ (in the BSS model the space $\tilde{Y}$ of guesses has to be R^∞; otherwise problems could be simplified; if, for example, $\tilde{Y} = \mathbb{Q}$, the decision problem $(\mathbb{R}, \mathbb{Q})$ becomes decidable by a BSS machine, contradicting the remark above). Then the machine must decide, in constant time, whether x, y belong to $\mathbb{N}$ or not. But this is impossible by Theorem 2. Therefore QDE is not $\mathbb{R}$-reducible to 3-SAT.)

3. A FURTHER REDUCTION TO $(F^2, F^2_{\text{zero},+})$: $NP_{\mathbb{R}}$-COMPLETENESS OF $(F^4, F^4_{\text{zero},+})$

Of course the main reason for the result in Theorem 2 and the corollary above is the different definitions of *size* in the two models. There are other problems changing their reducibility properties because of this. For example, Khachyan's algorithm for the linear programming problem (LP) is no longer a polynomial time algorithm in a real number model (cf. Traub and Wozniakowski, 1982) and it is conjectured that such an algorithm does not exist at all (cf. Traub and Wozniakowski, 1982; Schrijver, 1986).

This leads (because of the lemma below and under the hypothesis $P_{\mathbb{R}} \neq NP_{\mathbb{R}}$) to the *conjecture* "$(F^2, F^2_{\text{zero},+})$ does not belong to $P_{\mathbb{R}}$." Therefore, it probably is quite harder than (F^2, F^2_{zero}), which, in fact, belongs to $P_{\mathbb{R}}$ (see Triesch, 1989).

We regard the decision of solvability of an (LP)-problem in the equivalent "feasibility" form: given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, does there exist an $x \in \mathbb{R}^n$ s.t. $Ax \leq b$? The feasible problems constitute the set LP_{yes}.

LEMMA. $(LP, LP_{\text{yes}}) \propto_{\mathbb{R}} (F^2, F^2_{\text{zero},+})$.

Proof. Let $A =: [a_{ij}]$; then

$$\exists\, x \in \mathbb{R}^n \text{ s.t. } Ax \leq b \Leftrightarrow \exists\, x \in \mathbb{R}^n \text{ s.t. } Ax - b \leq 0$$

$$\Leftrightarrow \exists\, x \in \mathbb{R}^n \text{ s.t. } (A|-b)\binom{x}{1} \leq 0$$

$$\Leftrightarrow \exists\, x \in \mathbb{R}^n,\, y \in \mathbb{R},\, y > 0 \text{ s.t. } (A|-b)\binom{x}{y} \leq 0$$

$$\Leftrightarrow \exists\, \tilde{x} \in \mathbb{R}^{n+1} \text{ s.t. } \tilde{A}\cdot\tilde{x} \leq 0 \text{ and } c^T\cdot\tilde{x} > 0 \quad \text{with } c^T := (0, \ldots, 0, 1),\ \tilde{A} := (A|-b)$$

$\Leftrightarrow$ (Farkas' Lemma, see Schrijver, 1986) there do not exist $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\text{s.t.} \quad c = \sum_{i=1}^{m} \lambda_i \cdot \begin{pmatrix} a_i \\ -b_i \end{pmatrix} \qquad (a_i^T \text{ denotes the } i\text{th row vector of } A)$$

$$\Leftrightarrow f(\lambda_1, \ldots, \lambda_m) := \sum_{i=1}^{n} \left(\sum_{i=1}^{m} a_{ij}\lambda_i\right)^2 + \left(1 + \sum_{i=1}^{m} b_i\lambda_i\right)^2 \text{ does not have a nonnegative zero.}$$

But f is a polynomial of degree 2 with size polynomially bounded in m and n, and so we are done. ∎

Finally, we show that $(F^4, F^4_{\text{zero},+})$ is $NP_{\mathbb{R}}$-complete. This follows from the next theorem and the fact that (F^4, F^4_{zero}) is $NP_{\mathbb{R}}$-complete (see Blum *et al.*, 1989).

THEOREM 3. $(F^4, F^4_{\text{zero}}) \propto_{\mathbb{R}} (F^4, F^4_{\text{zero},+})$.

Proof. Let f: $\mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree 4; without loss of generality (cf. Blum *et al.*, 1989) we may assume

$$f(x_1, \ldots, x_n) = \left(t - \sum_{i=1}^{n} s_i x_i\right)^2 + \sum_{j=1}^{m} (x_{j,1} \cdot x_{j,2} - x_{j,3})^2,$$

where $t, s_i \in \mathbb{R}$, $x_{j,k} \in \{x_1, \ldots, x_n\}$. Define a polynomial $\tilde{f}$ of degree 8 as follows: for $1 \leq i \leq n$, let λ_i be a further unknown and

$$\tilde{f}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n) := \left(t - \sum_{i=1}^{n} s_i(\lambda_i - 1) \cdot x_i\right)^2$$

$$+ \sum_{j=1}^{m} [(\lambda_{j,1} - 1) \cdot (\lambda_{j,2} - 1) \cdot x_{j,1} \cdot x_{j,2} - (\lambda_{j,3} - 1) \cdot x_{j,3}]^2$$

$$+ \sum_{i=1}^{n} \lambda_i \cdot (\lambda_i - 2)^2$$

(here, if $x_{j,l} = x_k$ for some $k \in \{1, \ldots, n\}$, $\lambda_{j,l}$ means λ_k). Now suppose $(\tilde{x}_1, \ldots, \tilde{\lambda}_n)$ being a nonnegative zero of $\tilde{f}$. Then, in particular, $\tilde{\lambda}_i \in \{0, 2\}$, $1 \leq i \leq n$.

$$\textit{If } S' := \{i \mid \tilde{\lambda}_i = 0\}, \text{ let } y_i := \begin{cases} -\tilde{x}_i & i \in S' \\ \tilde{x}_i & i \notin S'. \end{cases}$$

It follows that $f(y_1, \ldots, y_n) = 0$.

Conversely, assume $f(y_1, \ldots, y_n) = 0$ for some $y_i \in \mathbb{R}$ and define

$$\tilde{\lambda}_i := \begin{cases} 0 & y_i < 0 \\ 2 & y_i \geq 0 \end{cases} \quad \text{and} \quad \tilde{x}_i := |y_i|.$$

Then we have $\tilde{x}_i \geq 0$, $\tilde{\lambda}_i \geq 0$, and $\tilde{f}(\tilde{x}_1, \ldots, \tilde{\lambda}_n) = 0$.

We still must reduce $\tilde{f}$ to a degree-4 polynomial g s.t. $\tilde{f}$ has a nonnegative zero iff g has one. The reduction can be done in the same way as in BSS; more explicitly: if $\tilde{f}(y) = \sum_{\alpha \in J} a_\alpha y^\alpha$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $\alpha_i \in \mathbb{N}$, let $t_\alpha := y^\alpha$ be new variables. Additionally, for each y_i (i.e., $\alpha = (0, \ldots, 0, 1, 0, \ldots, 0)$, 1 standing at position i) introduce a further unknown t_α, even if y_i does not appear as monomial in $\tilde{f}$. Then $\tilde{f}(y) = 0$ is equivalent to a system $\sum_{\alpha \in J} a_\alpha t_\alpha = 0$ together with finitely many equations of the type $t_{\alpha+\beta} = t_\alpha \cdot t_\beta$; finally, define g as the sum of the squares of this system (cf. Blum *et al.*, 1989, Sect. 4). Nonnegativity of zeros is transformed from one polynomial to the other because every unknown in $\tilde{f}$ also appears in g.

Again, all evaluations are polynomial time bounded in $\text{size}_{\mathbb{R}}(f)$. ■

Problem. What about the complexity of $(F^2, F^2_{\text{zero},+})$?

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